

Compressive Sensing Using the Entropy Functional

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Abstract

In most compressive sensing problems ℓ_1 norm is used during the signal reconstruction process. In this article the use of entropy functional is proposed to approximate the ℓ_1 norm. A modified version of the entropy functional is continuous, differentiable and convex. Therefore, it is possible to construct globally convergent iterative algorithms using Bregman's row action D-projection method for compressive sensing applications. Simulation examples are presented.

Index Terms

Compressive Sensing, Entropy functional, Iterative row-action methods, D-Projection.

I. INTRODUCTION

Nyquist-Shannon sampling theorem [1] is one of the fundamental theorems in signal processing literature. As it is well known, it specifies the conditions for perfect reconstruction of a continuous signal from its samples. If a signal is sampled with a rate at least two times its bandwidth, it can be perfectly reconstructed from its samples. However in many applications of signal processing including waveform compression, perfect reconstruction is not necessary.

The most common method used in compression applications is the transform coding. The signal $\mathbf{x}[n]$ is transformed into another domain defined by the transformation matrix ψ . The transformation procedure is simply finding the inner product of the signal $\mathbf{x}[n]$ with the rows ψ_i of the transformation matrix ψ as follows

$$s_i = \langle \mathbf{x}, \psi_i \rangle, \quad i = 1, 2, \dots, N, \quad (1)$$

where \mathbf{x} is a column vector, whose entries are samples of the signal $\mathbf{x}[n]$. The digital signal $\mathbf{x}[n]$ can be reconstructed from its transform coefficients s_i as follows;

$$\mathbf{x} = \sum_{i=1}^N s_i \cdot \psi_i \quad \text{or} \quad \mathbf{x} = \psi \cdot \mathbf{s} \quad (2)$$

where \mathbf{s} is a vector containing the transform domain coefficients, s_i . The basic idea in digital waveform coding is that the signal should be approximately reconstructed from only a few of its non-zero transform coefficients. In most cases including JPEG image coding standard, the transform matrix ψ is chosen such that the new signal \mathbf{s} is easily representable in the transform domain with a small number of coefficients. A signal \mathbf{x} is compressible, if it has a few large valued s_i coefficients in the transform domain and the rest of the coefficients are either zeros or very small valued.

In compressive sensing framework the signal is assumed to be a K-Sparse signal in a transformation domain such as wavelet domain or DCT domain. A signal with length N is K-Sparse, if it has K non-zero and $(N - K)$ zero coefficients in a transform domain. The case of interest in CS problems is when $K \ll N$ i.e., sparse in the transform domain.

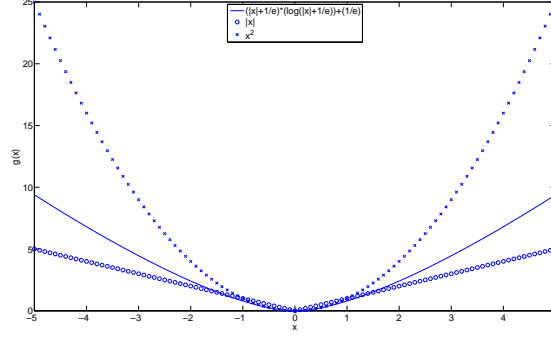


Fig. 1. Entropy functional $g(v)$ (—), $|v|$ (○) and the Euclidean cost function v^2 (×)

The CS theory introduced in [2]–[5] provides answers to the question of reconstructing a signal from its compressed measurements \mathbf{y} , which is defined as follows;

$$\mathbf{y} = \phi.\mathbf{x} = \phi.\psi.\mathbf{s} = \theta.\mathbf{s} \quad (3)$$

where ϕ is the $M \times N$ measurement matrix where $M \ll N$. The reconstruction of the original signal \mathbf{x} from its compressed measurements \mathbf{y} cannot be achieved by simple matrix inversion or inverse transformation techniques. A sparse solution can be obtained by solving the following optimization problem;

$$\mathbf{s}_p = \operatorname{argmin} \|\mathbf{s}\|_0 \quad \text{such that} \quad \theta.\mathbf{s} = \mathbf{y}. \quad (4)$$

However this problem is a NP-complete optimization problem therefore its solution can not be found easily. It is also shown in [2], [3] that, it is possible to construct the ϕ matrix from random numbers which are iid Gaussian random variables and choose the number of measurements as $cK \log(N/K) < M \ll N$ to satisfy the reconstruction conditions defined in [2] and [3]. With this choice of the measurement matrix, the optimization problem (4) can be approximated by ℓ_1 norm minimization as follows,

$$\mathbf{s}_p = \operatorname{argmin} \|\mathbf{s}\|_1 \quad \text{such that} \quad \theta.\mathbf{s} = \mathbf{y} \quad (5)$$

Entropy functional $g(v) = v \log v$ is used to approximately solve some ℓ_1 optimization problems and linear programming problems in signal and image reconstruction problems by Bregman [6] and others [13], [14], [18], [20] and [16]. In this article we propose the use of entropy functional as an alternative way to approximate the CS problem. In Figure 1, plots of the different cost functions including the shifted entropy function

$$g(v) = \left(|v| + \frac{1}{e}\right) \log\left(|v| + \frac{1}{e}\right) + \frac{1}{e}, \quad (6)$$

absolute value $g(v) = |v|$ and $g(v) = v^2$ are shown. The shifted entropy functional (6) is convex, continuous and differentiable, and it slowly increases compared to $g(v) = v^2$ because $\log(v)$ is much smaller than v for high v values as seen in Figure 1. Bregman also developed iterative row-action methods to solve the global optimization problem by successive local D-Projections. In each iteration step a D-projection, which is a generalized version of the orthogonal projection is performed

onto a hyperplane representing a row of the constraint matrix θ . In [6] Bregman proved that the proposed D-Projection based iterative method is guaranteed to converge to the global minimum regardless of the initial estimate.

An interesting interpretation of the row-action approach is that it provides a solution to the on-line CS problem. Each new measurement of the signal adds a row to the matrix θ . In the iterative row-action method a D-projection is performed to the new hyperplane formed by the new measurement. In this way, the currently available solution is updated without solving the entire CS problem. The new solution can be further updated using past measurements or new measurements in an iterative manner by performing other D-Projections. Therefore, it is possible to develop a real-time on-line CS method using the proposed approach.

The paper is organized as follows. In Section II, we review the D-Projection concept and define the modified entropy functional and related D-Projections. We generalized the entropy function based convex optimization method introduced by Bregman because the ordinary entropy function is defined only for positive real numbers. On the other hand, transform domain coefficients can be both positive and negative. The iterative CS algorithm is explained in Section II-A. We provide experimental results in Section III.

II. D-PROJECTION BASED ALGORITHM

The ℓ_o and ℓ_1 norm based cost functions (4) and (5) used in compressive sensing problems are not differentiable everywhere. In this article we use entropy functional based cost functions to find approximate solutions to the inverse problems defined in [2], [3]. Bregman developed convex optimization algorithms in 1960's and his algorithms are widely used in many signal reconstruction and inverse problems [9]–[15], [18]–[22].

Assume that the original signal \mathbf{x} can be represented by a K sparse length- N vector \mathbf{s} in a transform domain characterized by the transform matrix ψ . In CS problems the original signal \mathbf{x} is not available. However M measurements $\mathbf{y} = [y_1, y_2, \dots, y_M]^T = \phi\mathbf{x}$ of the original signal is observable via the measurement matrix ϕ , and the relation between \mathbf{y} and \mathbf{s} are described in Eq. (3).

Bregman's method provides globally convergent iterative algorithms for problems with convex, continuous and differentiable cost functionals $g(\cdot)$:

$$\min_{\mathbf{s} \in C} g(\mathbf{s}) \quad (7)$$

such that

$$\theta_i \cdot \mathbf{s} = y_i \quad \text{for } i = 1, 2, \dots, M. \quad (8)$$

where θ_i is the i -th row of the matrix θ . Each equation in (8) represents a hyperplane H_i in R^N , which are closed and convex sets in R^N . In Bregman's method the iterative reconstruction algorithm starts with an arbitrary initial estimate and successive D-projections are performed onto the hyperplanes H_i , $i = 1, 2, \dots, M$ in each step of the iterative algorithm.

The D-projection onto a closed and convex set is a generalized version of the orthogonal projection onto a convex set [6]. Let \mathbf{s}_o be arbitrary vector in R^N . Its D-Projection \mathbf{s}_p onto a closed convex set C with respect to a cost functional $g(\mathbf{s})$ is

defined as follows

$$\mathbf{s}_p = \arg \min_{\mathbf{s} \in C} D(\mathbf{s}, \mathbf{s}_o) \quad (9)$$

where

$$D(\mathbf{s}, \mathbf{s}_o) = g(\mathbf{s}_o) - g(\mathbf{s}) - \langle \nabla g(\mathbf{s}), \mathbf{s}_o - \mathbf{s} \rangle \quad (10)$$

In CS problems, we have M hyperplanes $H_i : \theta_i \cdot \mathbf{s} = y_i$ for $i = 1, 2, \dots, M$. For each hyperplane H_i , the the D-projection (9) is equivalent to

$$\nabla g(\mathbf{s}_p) = \nabla g(\mathbf{s}_o) + \lambda \theta_i \quad (11)$$

$$\theta_i \cdot \mathbf{s}_p = y_i \quad (12)$$

where λ is the Lagrange multiplier. As pointed above the D-projection is a generalization of the orthogonal projection. When the cost functional is the Euclidean cost functional $g(\mathbf{s}) = \sum_n s(n)^2$ the distance $D(\mathbf{s}_1, \mathbf{s}_2)$ becomes the ℓ_2 norm of difference vector $(\mathbf{s}_1 - \mathbf{s}_2)$, and the D-projection simply becomes the well-known orthogonal projection onto a hyperplane.

The orthogonal projection of an arbitrary vector $\mathbf{s}_o = [s_0[1], s_0[2], \dots, s_0[M]]$ onto the hyperplane H_i is given by

$$s_p(n) = s_o(n) + \lambda \theta_i(n), n = 1, 2, \dots, N \quad (13)$$

where $\theta_i(n)$ is the n -th entry of the vector θ_i and the Lagrange multiplier λ is given by,

$$\lambda = \frac{y_i - \sum_{n=1}^N s_o(n) \theta_i(n)}{\sum_{n=1}^N \theta_i^2(n)} \quad (14)$$

When the cost functional is the entropy functional $g(s) = \sum_n s(n) \log(s(n))$, the D-projection onto the hyperplane H_i leads to the following equations

$$s_p(n) = s_o(n) \cdot e^{(\lambda \cdot \theta_i(n))}, n = 1, 2, \dots, N \quad (15)$$

where the Lagrange multiplier λ is obtained by inserting (15) into the following hyperplane equation:

$$\theta_i \mathbf{s} = y_i \quad (16)$$

because the D-projection \mathbf{s}_p must be on the hyperplane H_i . The above set of equations are used in signal reconstruction from Fourier Transform samples [20] and the tomographic reconstruction problem [13]. The entropy functional is defined only for positive real numbers. In CS problems entries of vector \mathbf{s} can take both positive and negative values. We modify the entropy functional and extend it to negative real numbers as follows:

$$\min \sum_{i=1}^N \left(|s_i| + \frac{1}{e} \right) \cdot \left(\log \left(|s_i| + \frac{1}{e} \right) \right) \quad s.t. \quad \theta \cdot \mathbf{s} = y. \quad (17)$$

where subscript e represents the term entropy. The continuous cost functional $g_e(s)$ satisfies the following conditions,

- (i) $\frac{\partial g_e}{\partial s_i}(0) = 0$, $i = 1, 2, \dots, N$ and

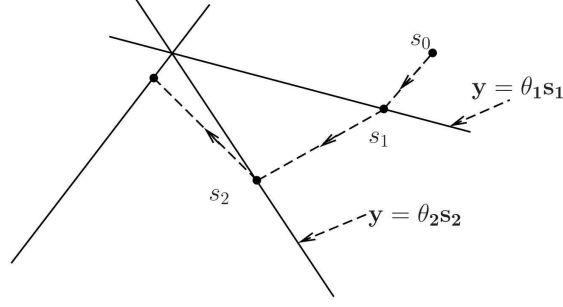


Fig. 2. Geometric interpretation of the entropic projection method: Sparse representation s_i corresponding to decision functions at each iteration are updated so as to satisfy the hyperplane equations defined by the measurements y_i and the measurement vector θ_i . Lines in the figure represent hyperplanes in \mathbb{R}^N . Sparse representation vector s_i converges to the intersection of the hyperplanes. Notice that D-projections are not orthogonal projections.

(ii) g_e is strictly convex and continuously differentiable.

On the other hand, the ℓ_1 norm is unfortunately not a globally smooth function therefore it can be solved using non differentiable minimization techniques such as sub-gradient methods [7]. Another way of approximating the ℓ_1 penalty function using an entropic functional is available in [8].

To obtain the D-projection of s_o onto a hyperplane H_i with respect to the entropic cost functional (17), we need to minimize the generalized distance $D(s, s_o)$ between s_o and the hyperplane H_i :

$$D(s, s_o) = g_e(s_o) - g_e(s) + \langle \nabla g_e(s), s - s_o \rangle \quad (18)$$

with the condition that $\theta_i s = y_i$. Using (12), the values of s can be obtained as;

$$s(n) = (s_o(n) + \text{sgn}(s(n)).e^{-1}).e^{(\text{sgn}(s(n)).\lambda.\theta_i(n))} - (\text{sgn}(s(n)).e^{-1}), \quad n = 1, 2, \dots, N. \quad (19)$$

where λ is the Lagrange multiplier.

The D-projection vector and the s_* satisfies the set of equations (19), and the hyperplane equation $H_i : \theta_i.s = y_i$.

A. Iterative Reconstruction Algorithm

The global convex optimization problem defined in (17) is solved by performing successive local D-projections onto hyperplanes defined by the rows of the matrix θ .

The iterations start with an arbitrary initial estimate s_o . This vector is D-projected onto the hyperplane H_1 and s_1 is obtained. The iterate s_1 is projected onto the next hyperplane H_2 (see Figure 2)... $N - 1^{st}$ estimate s_{N-1} is D-projected onto H_N and s_N is obtained. In this way the first iteration cycle is completed. The vector s_N is then projected onto the hyperplane H_1 and s_{N+1} is obtained etc. Bregman proved that s_i defined in (17) converges to the solution of the optimization problem.

III. EXPERIMENTAL RESULTS

For the validation and testing of the entropic minimization method, two experiments with two different signals are carried out. The *cusp* signal (Figure 3), which has 1024 samples, and the *random sparse* (Figure 4) signal, which has 128 samples, are used. The cusp signal is $S = 72$ sparse in DCT domain and the random signal has four non-zero samples.

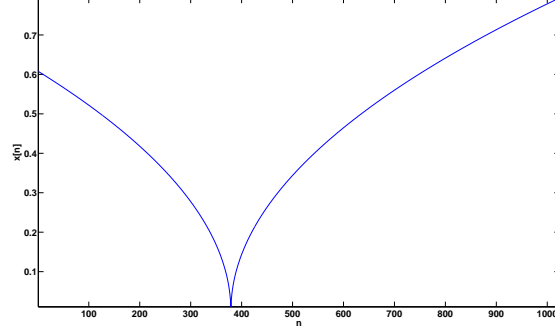


Fig. 3. The cusp signal with $N = 1024$ samples

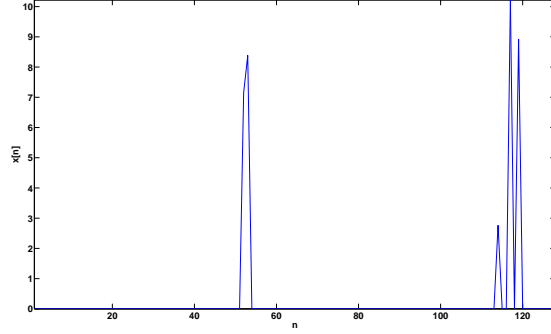


Fig. 4. Random $S=4$ sparse signal with $N = 128$ samples

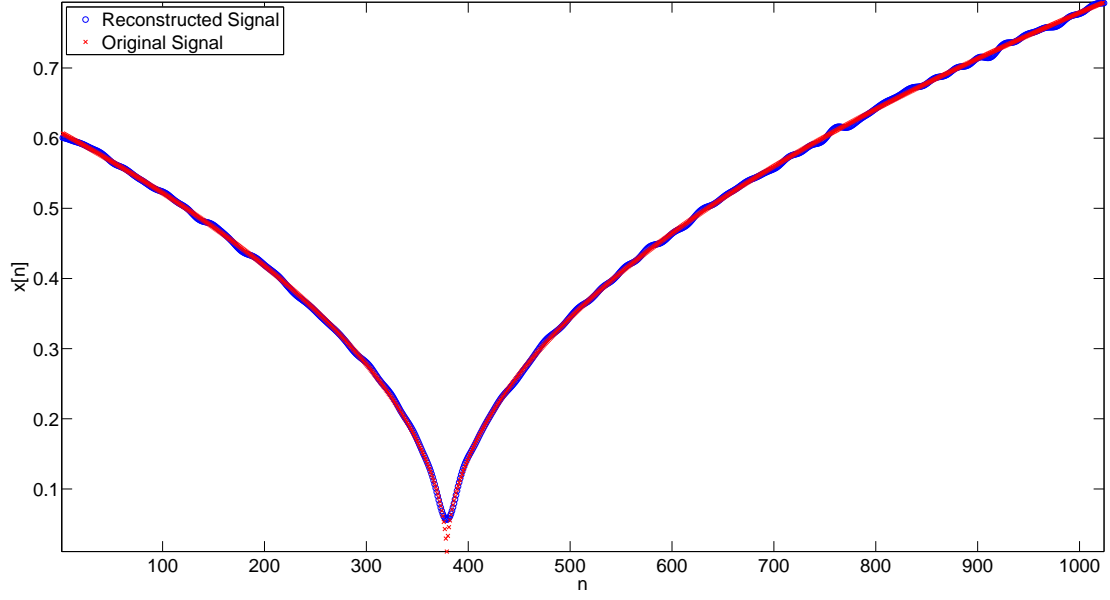
The measurement matrices ϕ are chosen as Gaussian random matrices. In the experiments $M = 2S$ and $M = 10S$ measurements are taken from the cusp signal and $M = 6S$ and $M = 10S$ measurements are taken from the random signal. The original signals are reconstructed from those measurements. The reconstructed signals using the iterative method using entropy based cost functional are shown in Figures 5(a), 5(b), 6(a), and 6(b). The reconstructed signals using the pseudo-inverse of the θ matrix are shown in Figures 7(a), and 7(b) respectively.

IV. CONCLUSION AND FUTURE WORK

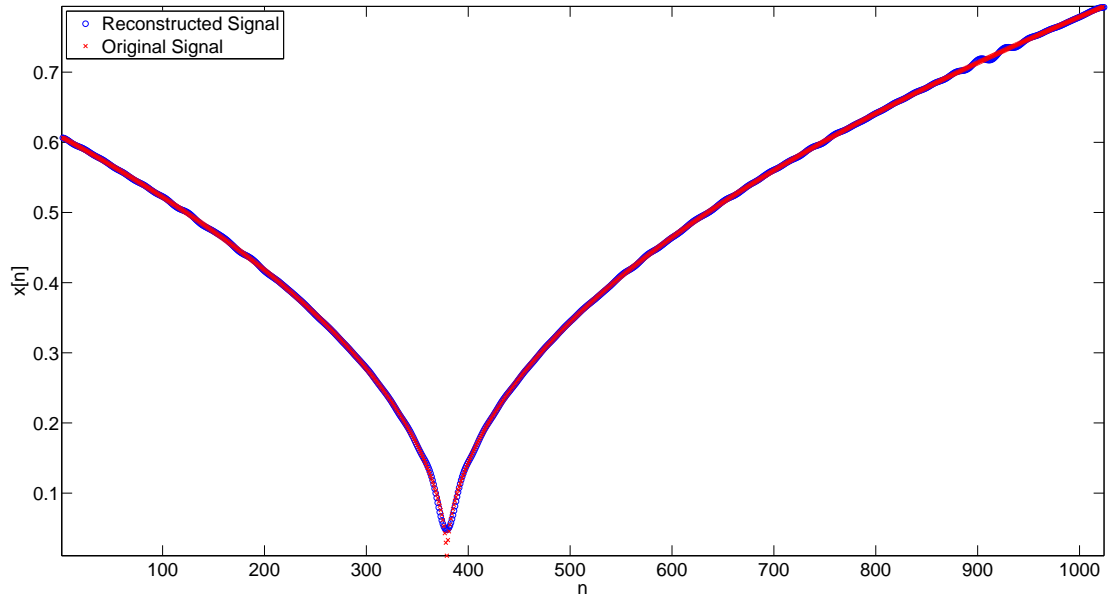
In this article we present; (i) the entropy based cost function for solving the CS problems, and (ii) an iterative row-action method to solve the CS problem. The entropy based cost functional estimates the ℓ_1 norm. Both the theoretical results given in Section II and the experimental results given in Section III indicate that the entropy based cost function and the iterative row-action method can be used for reconstructing a sparse signal from its measurements.

It is also shown that the row action methods provide a solution to the on-line CS problem. The reconstruction result can be updated on-line according to the new measurements without solving the entire optimization problem again in real time.

We will compare this entropy functional approach with other methods available including the TV approach [23], [24]

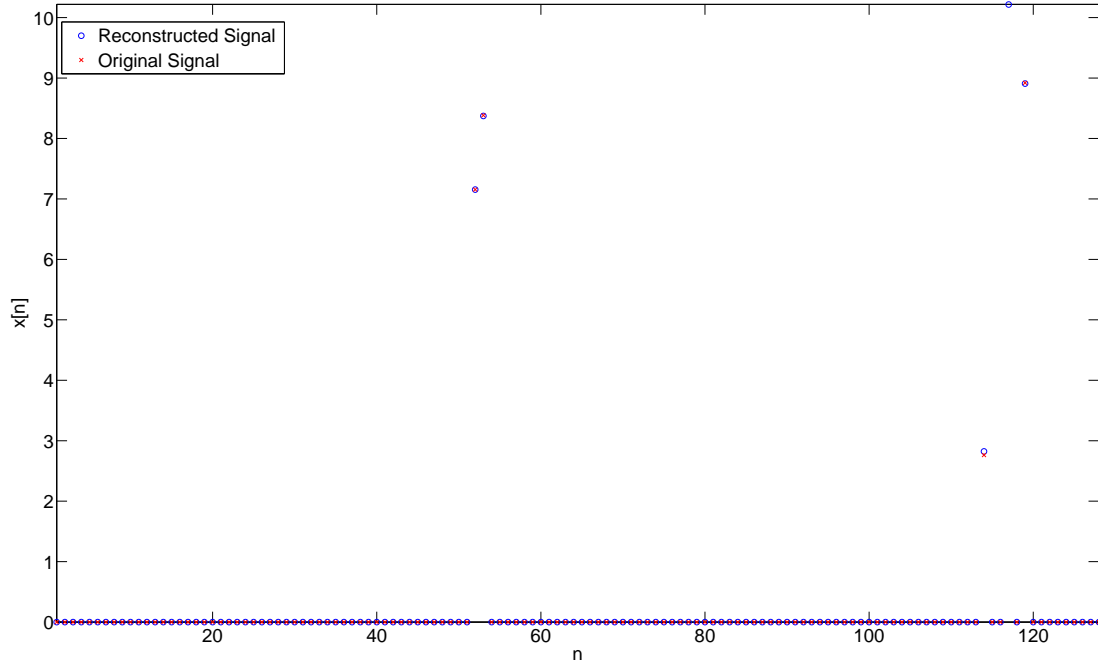


(a) $N = 1024$ length cusp signal reconstructed from $2S = 144$ measurements

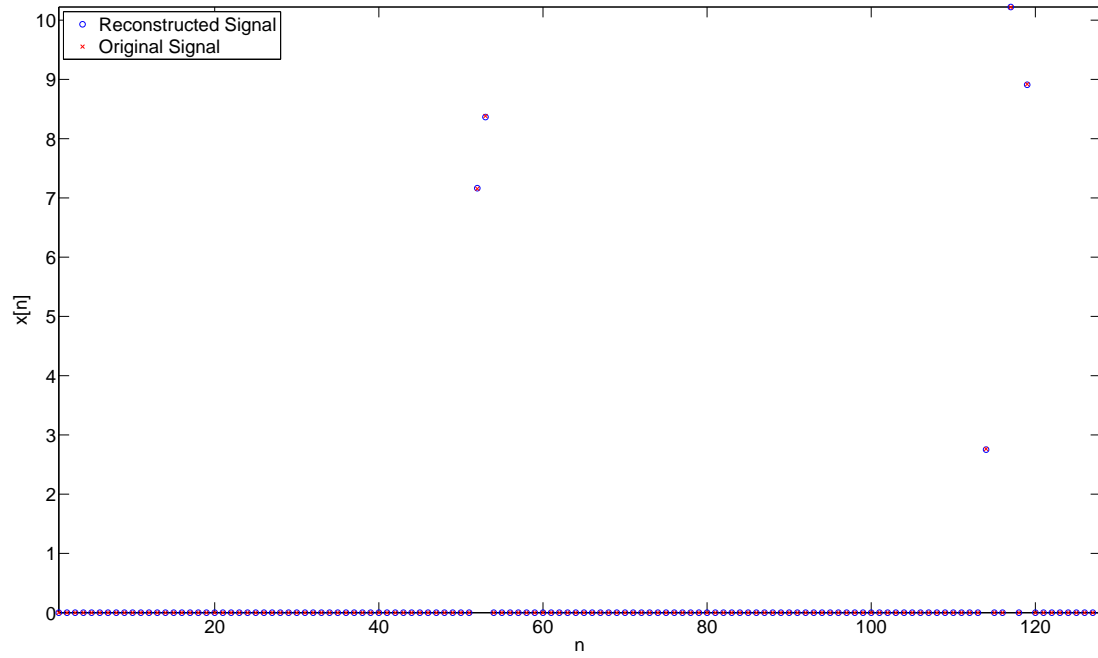


(b) $N = 1024$ length cusp signal reconstructed from $10S = 720$ measurements

Fig. 5. The cusp signal with 1024 samples reconstructed from $M = 2S$ (a) and $M = 10S$ (b) measurements using the iterative, entropy functional based method.

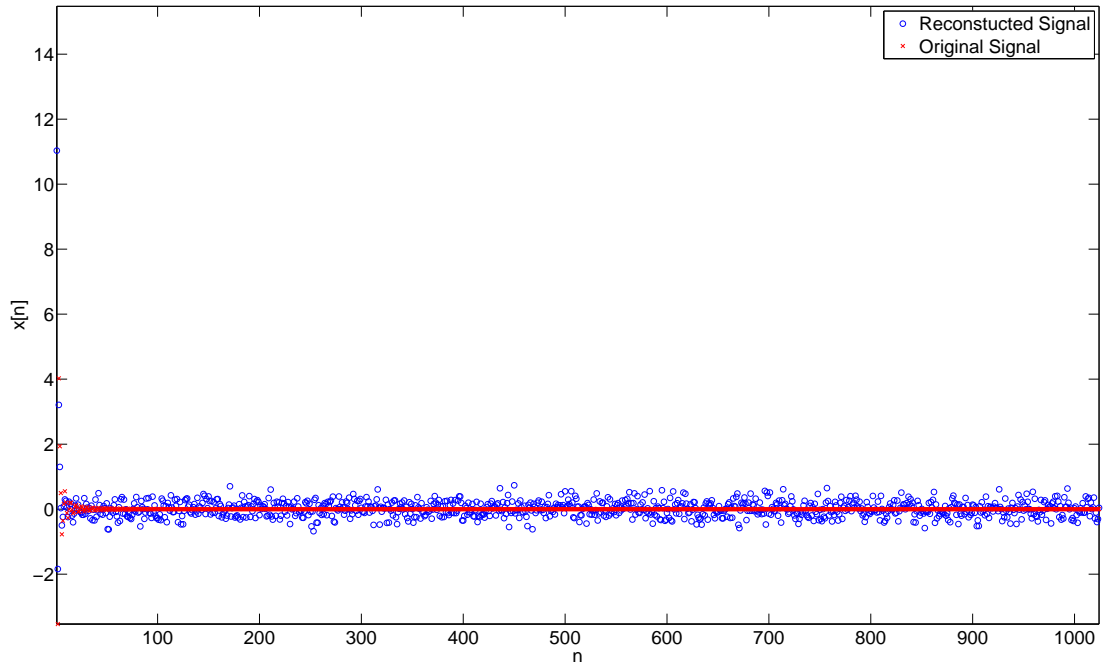


(a) $N = 128$ length random sparse signal reconstructed from $6S = 24$ measurements

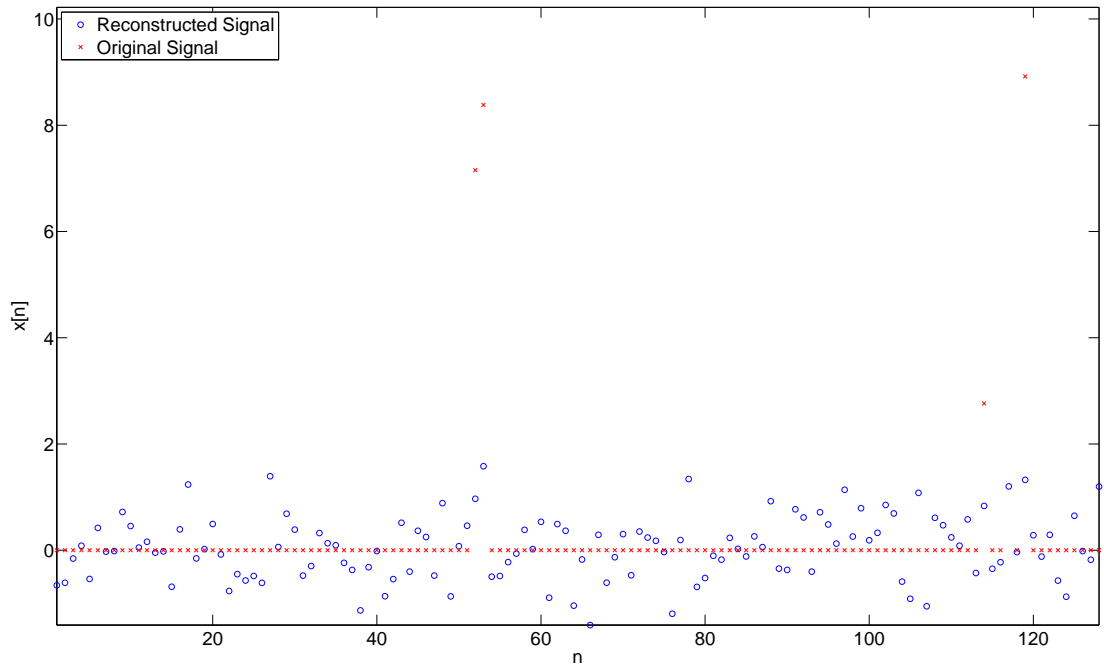


(b) $N = 128$ length random sparse signal reconstructed from $10S = 40$ measurements

Fig. 6. Random sparse signal with 128 samples is reconstructed from (a) $M = 6S$ and (b) $M = 10S$ measurements using the iterative, entropy functional based method.



(a) The DCT of the cusp signal reconstructed from its measurements using pseudo-inversion v.s. the DCT of the original cusp signal.



(b) Random sparse signal reconstructed from its measurements using pseudo-inversion v.s. the original signal.

Fig. 7. The DCT of the cusp signal and the random sparse signal reconstructed from their measurements using pseudo-inversion.

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